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Strong Asymptotics for the Continuous Sobolev Orthogonal Polynomials on the Unit Circle

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Strong (or Szegő-type) asymptotics for orthogonal polynomials with respect to a Sobolev inner product with general measures (the first measure is arbitrary and the second one is absolutely continuous and satisfying a smoothness condition) is obtained. Examples, illustrating the theorems proved, are presented.

Key Words: orthogonal polynomials; asymptotic properties; Sobolev inner product; measures on the unit circle.

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1. INTRODUCTION

Let $\{\tilde{Q}_n\}$ be the sequence of monic polynomials, $\deg \tilde{Q}_n = n$, which are orthogonal with respect to the Sobolev inner product on the unit circle $\Gamma = \{\xi \in \mathbb{C}: |\xi| = 1\}$,

$$\langle p, q \rangle_{W_2^1(\mu_0, \mu_1)} := \int_\Gamma p(\xi)\, \overline{q(\xi)}\, d\mu_0(\xi) + \int_\Gamma p'(\xi)\, \overline{q'(\xi)}\, d\mu_1(\xi), \tag{1}$$

where μ_i, $i = 0, 1$, are positive Borel measures supported on Γ and μ_1 is absolutely continuous with respect to the Lebesgue measure, and its Radon–Nikodym derivative is positive and smooth. In this paper, we study the strong (or Szegő-type) asymptotics of the polynomials orthogonal with respect to (1).

More precisely, we assume that

(i) μ_0 is an arbitrary positive Borel measure on Γ

(ii) μ_1 is absolutely continuous with respect to the Lebesgue measure, and (2)

$$\frac{d\mu_1(\xi)}{|d\xi|} = \rho_1(\xi) \quad \text{with} \quad \rho_1(\xi) > 0 \quad \text{for} \quad \xi \in \Gamma$$

and

$$\rho_1 \in C^{1+}(\Gamma).$$

The last condition in (2) means that the derivative of ρ_1 satisfies a Lipschitz condition with some positive exponent.

Until recently the asymptotic properties of general Sobolev orthogonal polynomials were considered only for the case when the measures corresponding to the derivatives in the Sobolev inner product were discrete. (For the case of Sobolev orthogonality on the interval $[a, b]$ see [3, 7, 8]; for the case when Γ is the unit circle see [6].)

At the VIII Symposium on Orthogonal Polynomials and their Applications [9], A. Martínez-Finkelshtein announced results on the strong asymptotics of Sobolev orthogonal polynomials with respect to measures supported on the interval $[a, b]$ and on the unit circle. These results generalize asymptotics of special Sobolev orthogonal polynomials on the interval with respect to special measures (see [11–13]). In another forthcoming paper (see [10]), the case of general smooth Jordan curves and arcs is studied. To our knowledge, the author has been able to prove strong asymptotics for Sobolev orthogonal polynomials on smooth curves when both measures are in Szegő's class. In the present paper, in comparison with [9] and [10], a bit more is required on the measure μ_1, and at the same time we obtain strong asymptotics even under very mild assumptions on the first measure μ_0.

In [1] and [2], asymptotic properties of Sobolev orthogonal polynomials on the unit circle, for the case when the measure μ_0 is general and μ_1 is the Lebesgue measure on the unit circle, $d\mu_1(\xi) = |d\xi|$, $\xi \in \Gamma$, were studied. The approach used there is easily generalized to the case when μ_1 is a Bernstein–Szegő measure. Standard techniques allow us to extend the study to the case when μ_1 is absolutely continuous and sufficiently smooth.

The following theorem holds true:

THEOREM 1. *Let* $\{\tilde{Q}_n\}$ *be the sequence of Sobolev orthogonal polynomials on the unit circle* Γ, *with respect to* (1), *with the measures* (μ_0, μ_1) *satisfying condition* (2). *Then, uniformly on compact subsets K of* $\Omega = \{\xi \in \mathbb{C} : |\xi| > 1\}$, *we have*

$$\frac{\tilde{Q}_n(z)}{z^n} \to F_1(z), \qquad \text{when} \qquad n \to \infty, \tag{3}$$

where F_1 *is the Szegő function corresponding to the weight* ρ_1. *That is,*

$$F_1(z) := \frac{D_1(z)}{D_1(\infty)}, \tag{4}$$

and $D_1(z)$ *verifies*

$$D_1, D_1^{-1} \in H(\Omega), \qquad D_1(\infty) > 0,$$

and

$$|D_1(\xi)|^2 = \frac{1}{\rho_1(\xi)}, \qquad \xi \in \Gamma.$$

Theorem 1 follows from Theorem 2 below where the asymptotics for the derivatives of polynomials $\tilde{Q}_n$ is obtained.

THEOREM 2. *Let* $\{\tilde{Q}_n'\}$ *be the sequence of derivatives of the Sobolev orthogonal polynomials with respect to the inner product* (1), *with measures satisfying condition* (2). *Then*

(i) $$\int_\Gamma \left| \frac{\tilde{Q}_n'(\xi)}{n\xi^{n-1}} - F_1(\xi) \right|^2 \rho_1(\xi)\, |d\xi| = o\left(\frac{1}{n}\right); \tag{5}$$

(ii) $$\frac{\tilde{Q}_n'(z)}{nz^{n-1}} \to F_1(z), \qquad n \to \infty, \quad \text{uniformly on} \quad z \in \bar{\Omega} = \{z \in \mathbb{C} : |z| \geqslant 1\}. \tag{6}$$

The proofs of Theorems 1 and 2 will be presented in the next two sections.

Now, we illustrate the theorems above with two examples of sequences of polynomials $\{\tilde{Q}_{n,1}\}$, $\{\tilde{Q}_{n,2}\}$ which are orthogonal on the unit circle with respect to the measures

$$\tilde{Q}_{n,1}: d\mu_0(\xi) = d\delta_a(\xi) \qquad \text{and} \qquad d\mu_1(\xi) = |d\xi|, \tag{7}$$

$$\tilde{Q}_{n,2}: d\mu_0(\xi) = |d\xi| + d\delta_a(\xi) \qquad \text{and} \qquad d\mu_1(\xi) = |d\xi| \tag{8}$$

(where δ_a is the Dirac delta measure with mass point at a, $|a| = 1$). These polynomials have explicit representations

$$\tilde{Q}_{n,1}(z) = z^n - a^n \qquad \text{for} \quad n \geqslant 1, \quad \tilde{Q}_{0,1}(z) = 1, \tag{9}$$

$$\tilde{Q}_{n,2}(z) = z^n - \frac{a^n H_{n-1}(z, a)}{1 + H_{n-1}(a, a)}, \tag{10}$$

where

$$H_{n-1}(z, y) = \sum_{k=0}^{n-1} \frac{z^k \bar{y}^k}{1+k^2}.$$

Indeed, it is easy to verify that for $n \geqslant 1$ and $k = 0, ..., n$,

$$\int_\Gamma \tilde{Q}_{n,1}(\xi)\, \overline{\xi^k}\, d\delta_a(\xi) + \int_\Gamma \tilde{Q}'_{n,1}(\xi)\, \overline{k\xi^{k-1}}\, |d\xi|$$

$$= \tilde{Q}_{n,1}(a)\, \bar{a}^k + \int_\Gamma n\xi^{n-1}\, \overline{k\xi^{k-1}}\, |d\xi| = n^2 \delta_{n,k},$$

and for $n = 0$, $\int_\Gamma \tilde{Q}_{0,1}(\xi)\, d\delta_a(\xi) = 1$.

On the other hand, it is known that the orthogonal polynomial sequence with respect to the inner product (1) with $d\mu_0(\xi) = d\mu_1(\xi) = |d\xi|$ is $\{z^n\}$, with $\|z^n\|^2_{W_2^1(\mu_0, \mu_1)} = n^2 + 1$ and the nth kernel function is given by $H_n(z, y)$ defined above. Moreover, the Sobolev inner product (8) can be considered as the modification of the previous one by adding a Dirac measure at a. Therefore, we obtain (10), (see [4, p. 38]).

From this representation, it follows directly that for both families of polynomials (9) and (10), the asymptotics (3) and (6) hold true with $F_1 = 1$. At the same time, the asymptotic formula (3) for these polynomials cannot be extended uniformly up to the boundary of Ω (i.e., uniformly on the unit circle Γ). Indeed, for (9) this is obvious, and for (10), we have

$$\left| \frac{\tilde{Q}_{n,2}(e^{i\varphi} a)}{(e^{i\varphi} a)^n} - 1 \right| = \frac{\displaystyle\sum_{k=0}^{n-1} \frac{(e^{i\varphi})^n}{1+k^2}}{1 + \displaystyle\sum_{k=0}^{n-1} \frac{1}{1+k^2}}.$$

So, at least for $\varphi = 0, \pi/2, \pi$, and $3\pi/2$, we see that

$$\lim_{n \to \infty} \frac{\tilde{Q}_{n,2}(\xi)}{\xi^n} \neq 1 \qquad \text{for} \quad \xi = e^{i\varphi} a.$$

The last circumstance is particularly interesting because the polynomials (9) and (10) are uniformly bounded on Γ. Again, for the polynomials $\tilde{Q}_{n,1}$ it is trivial whereas

$$|\tilde{Q}_{n,2}(\xi)| \leqslant 1 + \frac{\sum_{k=0}^{n-1} \frac{1}{1+k^2}}{1 + \sum_{k=0}^{n-1} \frac{1}{1+k^2}} < 2, \qquad \forall n, \forall \xi \in \Gamma.$$

Thus, uniform boundedness of the polynomials on Γ and smooth boundary values of $F_1(z)$ are not sufficient conditions for solving the Tauberian problem (see [4, p. 75]), of extending the asymptotic relation (3) up to the boundary Γ of Ω. Perhaps more restrictive conditions on the measure μ_0 are needed.

2. PROOF OF THEOREM 2

Let $\{Q_n\}$ denote the monic orthogonal polynomial sequence with respect to measure μ_1. We recall an extremal property of monic orthogonal polynomials with respect to the inner products in $W_2^1(\mu_0, \mu_1)$ and in $L_2(\mu_1)$,

$$\|\tilde{Q}_n\|^2_{(\mu_0,\mu_1)} := \langle \tilde{Q}_n, \tilde{Q}_n \rangle_{W_2^1(\mu_0,\mu_1)} = \min_{P_n} \|P_n\|^2_{(\mu_0,\mu_1)}, \tag{11}$$

$$\|Q_n\|^2_{\mu_1} := \int_\Gamma |Q_n|^2 \rho_1(\xi)\, |d\xi| = \min_{P_n} \|P_n\|^2_{\mu_1}, \tag{12}$$

for any monic polynomial $P_n(\xi) = \xi^n +$ lower degree terms. Under the conditions in (2) on measure μ_1, the following asymptotics for the sequence of orthogonal polynomials $\{Q_n\}$ hold (see [14]),

$$\|Q_n\|^2_{\mu_1} = m_1 + o\left(\frac{1}{n}\right), \tag{13}$$

and uniformly on $\bar{\Omega}$

$$\left|\frac{Q_n(z)}{z^n} - F_1(z)\right| = o\left(\frac{1}{n}\right), \qquad z \in \bar{\Omega}, \tag{14}$$

where F_1 is defined as in (4) and

$$m_1 = \frac{2\pi}{D_1^2(\infty)} = \|F_1\|_{\mu_1}^2.$$

Also under the conditions in (2), (see [5]), the asymptotic formula can be differentiated as

$$\max_{z \in \bar{\Omega}} \left| \frac{d}{dz} \left(\frac{Q_n(z)}{z^n} - F_1(z) \right) \right| = o(1). \tag{15}$$

Moreover, from (13) and (15), it follows that

$$\frac{\|Q'_n\|_{\mu_1}^2}{n^2} = m_1 + o\left(\frac{1}{n}\right). \tag{16}$$

Indeed, to see that (16) holds true, we take into account that from (12) and (13), we immediately have an estimate of (16) from below. That is,

$$m_1 + o\left(\frac{1}{n}\right) = \|Q_{n-1}\|_{\mu_1}^2 \leqslant \frac{\|Q'_n\|_{\mu_1}^2}{n^2}.$$

An estimate from above can be obtained by substituting uniform asymptotics (15) on Γ into the left-hand side of (16) and taking into account that under condition (2) the Szegő function is smooth on Γ (see [16], Lemma 4.1):

$$F_1(\xi) \in C^{1+}(\bar{\Omega}). \tag{17}$$

Indeed, from (15) it follows

$$\left\| \frac{Q'_n(\xi)}{\xi^n n} \right\|_{\mu_1}^2 \leqslant \left\| \frac{Q_n(\xi)}{\xi^{n-1}} + \frac{F'_1(\xi)}{n} \right\|_{\mu_1}^2 + o\left(\frac{1}{n^2}\right),$$

and we obtain (16) because of

$$\left\| \frac{Q_n(\xi)}{\xi^{n-1}} + \frac{F'_1(\xi)}{n} \right\|_{\mu_1}^2 \leqslant \left\| \frac{Q_n(\xi)}{\xi^{n-1}} \right\|_{\mu_1}^2 + \left\| \frac{F'_1(\xi)}{n} \right\|_{\mu_1}^2 = m_1 + o\left(\frac{1}{n}\right) + O\left(\frac{1}{n^2}\right).$$

The same asymptotics are valid for the norms of the Sobolev orthogonal polynomials.

LEMMA 1.

$$\text{(i)} \qquad \frac{\|\tilde{Q}_n'\|^2_{\mu_1}}{n^2} = m_1 + o\left(\frac{1}{n}\right); \tag{18}$$

$$\text{(ii)} \qquad \frac{\|\tilde{Q}_n\|^2_{(\mu_0, \mu_1)}}{n^2} = m_1 + o\left(\frac{1}{n}\right). \tag{19}$$

Proof. From the extremal properties (11) and (12), we have

$$m_1 + o\left(\frac{1}{n}\right) = \|Q_{n-1}\|^2_{\mu_1} \leqslant \frac{\|\tilde{Q}_n'\|^2_{\mu_1}}{n^2} \leqslant \frac{\|\tilde{Q}_n\|^2_{(\mu_0, \mu_1)}}{n^2}$$

$$\leqslant \frac{\|Q_n\|^2_{\mu_0}}{n^2} + \frac{\|Q_n'\|^2_{\mu_1}}{n^2}.$$

Because of (14), the sequence $\{Q_n\}$ is uniformly bounded, and by (16) we conclude

$$\frac{\|Q_n\|^2_{\mu_0}}{n^2} + \frac{\|Q_n'\|^2_{\mu_1}}{n^2} = O\left(\frac{1}{n^2}\right) + m_1 + o\left(\frac{1}{n}\right).$$

Thus, the lemma is proved. ∎

Proof of Theorem 2. The validity of the theorem follows from Lemma 1 using standard method. For assertion (i) (see (5)), we have

$$\left\| \frac{\tilde{Q}_n'(\xi)}{n\xi^{n-1}} - F_1(\xi) \right\|^2_{\mu_1} = \left\| \frac{\tilde{Q}_n'(\xi)}{n} \right\|^2_{\mu_1} + \|F_1(\xi)\|^2_{\mu_1}$$

$$- 2\Re \int_\Gamma F_1(\xi) \overline{\left(\frac{\tilde{Q}_n'(\xi)}{n\xi^{n-1}}\right)} \rho_1(\xi)\, |d\xi|.$$

The reproducing property of the Szegő function in the Hardy space $H^2_{\rho_1}(\Omega)$ (see, for example, [16]) states that

$$\forall h(z) \in H^2_{\rho_1}(\Omega) \Rightarrow h(\infty) = \int_\Gamma \frac{F_1(\xi)}{m_1} \overline{h(\xi)}\, \rho_1(\xi)\, |d\xi|.$$

Therefore, the integral in the equality above is equal to m_1. Thus, (15) and (18) give us (5).

Assertion (ii) (see (6)) follows from (5). In fact, let us denote

$$\mathfrak{F}_n(\xi) := \frac{\tilde{Q}_n'(\xi)}{n} - \xi^{n-1} F_1(\xi), \qquad \text{and} \qquad A_n := \max_\Gamma |\mathfrak{F}_n(\xi)|.$$

From the continuity of F_1 on Γ and the definition of A_n, we have

$$\max_{\Gamma} |\tilde{Q}_n'(\xi)| = O(n(A_n+1)).$$

Now, applying the Markov–Bernstein–Szegő inequality to the second derivative of $\tilde{Q}_n$, (see, for example, Lemma 2.1 in [16]) we obtain

$$\max_{\Gamma} \left| \frac{d}{d\xi} \mathfrak{F}_n(\xi) \right| \leqslant \frac{1}{n} \max_{\Gamma} \left| \frac{d}{d\xi} \tilde{Q}_n'(\xi) \right| + \max_{\Gamma} \left| \frac{d}{d\xi} (\xi^{n-1} F_1(\xi)) \right| \leqslant an(A_n+1), \tag{20}$$

for some positive constant a. Assume that ξ_0 is such that $A_n = |\mathfrak{F}_n(\xi_0)|$ and let $e \subset \Gamma$ be an arc centered at $\xi_0 \in e$ with $|e| = A_n/an(A_n+1)$. For any $\xi \in e$, from (20) we have that

$$|\mathfrak{F}_n(\xi_0)| - |\mathfrak{F}_n(\xi)| \leqslant \frac{A_n}{2},$$

which implies $A_n/2 \leqslant |\mathfrak{F}_n(\xi)|$. Therefore, taking into account of (5), it follows that

$$\frac{A_n^3}{4an(A_n+1)} \min_{\Gamma} \rho_1 \leqslant \int_e |\mathfrak{F}_n(\xi)|^2 \rho_1(\xi) \, |d\xi|$$

$$\leqslant \int_{\Gamma} |\mathfrak{F}(\xi)|^2 \rho_1(\xi) \, |d\xi| = o\left(\frac{1}{n}\right).$$

Therefore, $A_n \to 0$ as $n \to \infty$. This, in turn, implies that

$$\frac{\tilde{Q}_n'(\xi)}{n\xi^{n-1}} \to F_1(\xi) \qquad \text{uniformly for} \quad \xi \in \Gamma, \quad n \to \infty.$$

Now, if we apply the Maximum Modulus Principle for analytic functions, we deduce (6) and thus the theorem is proved. ∎

3. PROOF OF THEOREM 1

First, we obtain some auxiliary bounds on Γ for the Sobolev orthogonal polynomials with respect to (1), with assumptions (2).

LEMMA 2. *Let* $\{\tilde{Q}_n\}$ *be the sequence of monic orthogonal polynomials defined by* (1) *and* (2). *Then*

$$\max_{\Gamma} |\tilde{Q}_n(\xi)| \leqslant O(n).$$

Proof. Let ξ_0 be an arbitrary point on Γ and let $\xi_{1,n}$ be the point where

$$|\tilde{Q}_n(\xi_{1,n})| = \min_{\xi \in \Gamma} |\tilde{Q}_n(\xi)|,$$

then

$$|\tilde{Q}_n(\xi_{1,n})|^2 \int_{\Gamma} d\mu_0(\xi) \leqslant \int_{\Gamma} |\tilde{Q}_n(\xi)|^2 \, d\mu_0(\xi) \leqslant n^2 m_1 + o(n).$$

Notice that the last inequality follows from (19). Therefore,

$$|\tilde{Q}_n(\xi_{1,n})|^2 \leqslant \mathrm{O}(n^2).$$

From (6), we have

$$\max_{\Gamma} |\tilde{Q}'_n(\xi)| < \mathrm{O}(n). \tag{21}$$

Then,

$$|\tilde{Q}_n(\xi_0)| \leqslant |\tilde{Q}_n(\xi_{1,n})| + 2\pi \max_{\Gamma} |\tilde{Q}'_n(\xi)| \leqslant \mathrm{O}(n). \tag{22}$$

Now, let $\tilde{\xi}_n$ be such that $|\tilde{Q}_n(\tilde{\xi}_n)| = \max_{\Gamma} |\tilde{Q}_n(\xi)|$. Thus, (21) and (22) give us

$$|\tilde{Q}_n(\tilde{\xi}_n)| \leqslant \left| \int_{\xi_0}^{\tilde{\xi}_n} \tilde{Q}'_n(\xi) \, d\xi \right| + |\tilde{Q}_n(\xi_0)| \leqslant \mathrm{O}(n).$$

With this we conclude the proof. ∎

From Lemma 2 and the Maximum Modulus Principle, we obtain:

COROLLARY 1. *For any compact* $K \subset \Omega$, *there exists a constant* $C(K)$, *such that*

$$\max_{z \in K} \left| \frac{\tilde{Q}_n(z)}{z^n} \right| < C(K) \, n.$$

Proof of Theorem 1. Let K be a compact subset of Ω and let z be an arbitrary point in K. There exists r_0 such that $1 < r_0 < |z|$, for all $z \in K$. Choose z_0 in the segment joining z and 0 such that $|z_0| = r_0$. Let γ be a segment joining z_0 and z, where $|\xi| \leqslant |z|$, for all $\xi \in \gamma$. Then

$$\frac{\tilde{Q}_n(z)}{z^n} = \frac{1}{z^n} \int_\gamma \tilde{Q}'_n(\xi)\, d\xi + \frac{\tilde{Q}_n(z_0)}{z_0^n} \left(\frac{z_0}{z}\right)^n.$$

From Corollary 1, it follows that the last term tends uniformly to zero. Applying Theorem 2 and using integration by parts, we have

$$\begin{aligned} \frac{1}{z^n} \int_\gamma \tilde{Q}'_n(\xi)\, d\xi &= \frac{1}{z^n} \int_{z_0}^{z} n\xi^{n-1}(F_1(\xi) + o(1))\, d\xi \\ &= \frac{1}{z^n} [z^n(F_1(z) + o(1))] \\ &\quad - \frac{1}{z^n} [z_0^n(F_1(z_0) + o(1))] \\ &\quad - \frac{1}{z^n} \int_{z_0}^{z} \xi^n \frac{d}{d\xi} (F_1(\xi) + o(1))\, d\xi. \end{aligned}$$

Here, the second term tends uniformly to zero as $n \to \infty$ and the last integral is estimated as follows. If $z = re^{i\varphi}$ and $z_0 = r_0 e^{i\varphi}$, then

$$\begin{aligned} &\frac{1}{|z|^n} \left| \int_{z_0}^{z} \xi^n \frac{d}{d\xi} (F_1(\xi) + o(1))\, d\xi \right| \\ &\leqslant \frac{1}{r^n} \max_\gamma \left| \frac{d}{d\xi} F_1(\xi) \right| \left| \int_{r_0}^{r} s^n\, ds \right. \\ &= \max_\gamma \left| \frac{d}{d\xi} F_1(\xi) \right| \left(\frac{r}{(n+1)} - \frac{r_0}{(n+1)} \left(\frac{r_0}{r}\right)^n \right) \to 0, \qquad n \to \infty. \end{aligned}$$

The theorem is proved. ∎

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